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Some properties of the Calogero–Sutherland model with reflections

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Abstract. We prove that the Calogero–Sutherland model with reflections (the BC_N model) possesses a property of duality relating the eigenfunctions of two Hamiltonians with different coupling constants. We obtain a generating function for their polynomial eigenfunctions, the generalized Jacobi polynomials. The symmetry of the wavefunctions for certain particular cases (associated with the root systems of the classical Lie groups B_N, C_N and D_N) is also discussed.

1. Introduction

During the last few years a lot of work has been devoted to the study of the Calogero–Sutherland Hamiltonian because of its relation to fractional statistics in one dimension, to the random matrix theory, etc. The model originally proposed by Sutherland [1] describes particles moving on a circle, with an interaction proportional to the inverse square of the chord distance and with periodic boundary conditions. This Hamiltonian will be referred to as the periodic Calogero–Sutherland Hamiltonian. This model is exactly solvable and its wavefunctions, the Jack polynomials, have been extensively studied [2, 3].

A new family of models of the Calogero–Sutherland type was proposed in [4], describing particles on a semicircle, interacting with one another and with the boundaries. We call them Calogero–Sutherland models with reflections; several types are associated with several types of root systems of classical Lie algebras. These models are also exactly solvable. Some properties of their polynomial eigenfunction, the Macdonald polynomials, were studied in [5]. The spectrum and the eigenfunctions of these Hamiltonians were used in [6] in order to obtain the exact solution of a class of long-range interacting spin chains with boundaries.

These models are remarkably similar to the periodic one, but there are additional complications related to the loss of the translational invariance. One of the key properties of the periodic model is the duality which permits relating the wavefunctions corresponding to two different values of the coupling constant [2, 3, 7]. In this paper we prove the existence of a similar duality property for the Calogero–Sutherland Hamiltonian with reflections. The initial motivation of this work was in obtaining the correlation functions of the Calogero–Sutherland models with reflections.

The plan of the paper is the following: the next section is devoted to the presentation of the model, in section 3 we present a method for deriving the polynomial eigenfunctions and in the section 4 we emphasize the connection between the eigenfunctions of these

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Hamiltonians and the Jacobi functions. Section 5 is devoted to the proof of the duality property. In section 6 we use this property in order to derive an expansion formula for the kernel which intertwines between the two dual models.

2. The model

Following [4, 5], a Hamiltonian of Calogero–Sutherland (CS) type can be defined for each root system of a classical Lie algebra. Let V denote on N-dimensional vector space with an orthonormal basis $\{e_1, \ldots, e_N\}$ and let $R = \{\alpha\}$ be a root system in V, with R_+ the set of positive roots. Let θ denote the vector $(\theta_1, \ldots, \theta_N)$ and $\theta \cdot \alpha$ its scalar product with the vector α. The generalized CS Hamiltonian is

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial\theta_i^2} + \sum_{\alpha\in R_+} \frac{g_\alpha}{\sin^2(\theta\cdot\alpha/2)} \tag{2.1}$$

where g_α is constant on each orbit of the Weyl group, i.e. it has the same value for the roots of the same length.

The periodic model [1] corresponds with the root system of type A_{N-1}. It describes interacting particles on a circle, with the positions specified by angles θ_i ranging from 0 to 2π and with periodic boundary conditions.

The reflection models are associated with the four infinite series of root systems D_N, B_N, C_N and BC_N. A list of the main characteristics of these series of root systems is given in appendix A.

The most general Hamiltonian is the one associated with the *non-reduced* (i.e. that includes roots which are proportional, as α and 2α) BC_N root system; the others can be obtained from it by setting the coupling constants to some special values.

The BC_N Hamiltonian describes N particles on a semicircle, with positions specified by the angles $0 \leqslant \theta_1, \ldots, \theta_N \leqslant \pi$:

$$\begin{aligned} H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial\theta_i^2} + \beta(\beta-1) \sum_{i\neq j=1}^{N} \left[\sin^{-2}\left(\frac{\theta_i-\theta_j}{2}\right) + \sin^{-2}\left(\frac{\theta_i+\theta_j}{2}\right)\right] \\ + \sum_{i=1}^{N} \left[c_1(2c_2+c_1-1)\sin^{-2}\frac{\theta_i}{2} + 4c_2(c_2-1)\sin^{-2}\theta_i\right]. \end{aligned} \tag{2.2}$$

This Hamiltonian has three independent coupling constants β, c_1 and c_2, corresponding to the roots of length 2, 1 and 4, respectively.

Compared with the periodic version, the potential part of this Hamiltonian contains a new two-body term corresponding to the interaction between particle i and the 'image' of the particle j through the reflection $\theta_j \to -\theta_j$. Using the relation $\sin^{-2} x + \cos^{-2} x = 4\sin^{-2}(2x)$, the one-body part of the potential can be separated into couplings of the particles to the two boundaries $\theta = 0, \pi$, with independent coupling constants related to c_1, c_2.

The other cases are obtained by setting to zero one (or both) of the coupling constants c_1, c_2:

$$B_N:\ c_2 = 0 \qquad C_N:\ c_1 = 0 \qquad D_N:\ c_1 = c_2 = 0. \tag{2.3}$$

The symmetry to be imposed to the wavefunctions depends on the root system we consider.

The ground-state wavefunction of this Hamiltonian is [6]

$$\Delta(\theta) = \prod_{i=1}^{N} \left[\sin^{c_1}\frac{\theta_i}{2}\sin^{c_2}\theta_i\right] \prod_{i<j} \left[\sin^{\beta}\left(\frac{\theta_i-\theta_j}{2}\right)\sin^{\beta}\left(\frac{\theta_i+\theta_j}{2}\right)\right]. \tag{2.4}$$

Note that this ground state is well defined for $\beta, c_1, c_2 > -1/2$.

It is convenient to define a gauge transformed Hamiltonian by $\mathcal{H} = \Delta(\theta)^{-1} H \Delta(\theta) - E_0$, with $E_0 = \sum_{i=1}^{N} (\beta(N-i) + c_1/2 + c_2)^2$ the ground-state energy of H. We obtain

$$\mathcal{H} = -\sum_{i=1}^{N} \partial_i^2 - \beta \sum_{i \neq j} \left[\operatorname{ctg}\left(\frac{\theta_i - \theta_j}{2} \right) + \operatorname{ctg}\left(\frac{\theta_i + \theta_j}{2} \right) \right] \partial_i - \sum_{i=1}^{N} \left[c_1 \operatorname{ctg} \frac{\theta_i}{2} + 2c_2 \operatorname{ctg} \theta_i \right] \partial_i . \tag{2.5}$$

Let us mention that higher-order conserved quantities exist for this model. They are generated by the quantum determinant of the monodromy matrix obeying the reflection equation [8]. Their construction parallels that of the conserved quantities of the periodic model [9]. The monodromy matrix for the spin chains associated with this model was constructed in [6].

3. Symmetry of the eigenstates of $\mathcal{H}$

In this section we present the basis of the polynomials in the variables $z_j^{\pm 1/2} = \mathrm{e}^{\pm \mathrm{i}\theta_j/2}$ in which the Hamiltonian $\mathcal{H}$ is triangular [5]. This basis can serve for determining the eigenvalues and for finding the eigenfunctions.

We emphasize that different symmetries can be assigned to these eigenfunctions, depending on the values of the coupling constants c_1, c_2. These symmetries can be best understood in terms of root systems [5], as invariances under transformations defined by the Weyl group. The wavefunctions are naturally indexed by the dominant weights of the root systems BC_N (or D_N, B_N, C_N for the particular values of the coupling constants mentioned in (2.3)).

We start this section with a brief review (for more details see, for example, [10]) of some of the notions related to the root systems.

Let V be an N-dimensional vector space with an orthonormal basis $\{e_1, \ldots, e_N\}$ and α a root system in V. Let X be the reunion of hyperplanes orthogonal to one of the roots α. A chamber is a connected component of $V - X$. Let $(\alpha_1, \ldots, \alpha_l)$ be the basis corresponding to a chamber C ($(\alpha_i, x) > 0$ for $x \in C$) and $\alpha_i^V = 2\alpha_i/(\alpha_i, \alpha_i)$. The vectors $\bar{\omega}_i$ with the property $(\alpha_i^V, \bar{\omega}_j) = \delta_{ij}$ are called the fundamental weights. The dominant weights are defined as being linear combinations of the fundamental weights with non-negative integer coefficients, $\lambda = \sum_{i=1}^{l} k_i \bar{\omega}_i$. When $l = N$, as in the cases considered here, the dominant weights can equally be characterized by the set of coordinates $\lambda_1, \ldots, \lambda_N$ of λ with respect to the the orthogonal system $e_1, \ldots, e_N$; $\lambda_i = (\lambda, e_i)$.

We note that the Weyl group is the group generated by the reflections with respect to the hyperplanes orthogonal to the roots.

The main characteristics of the root systems we consider, as well as the allowed values of $\lambda_1, \ldots, \lambda_N$, are presented in appendix A.

A *partial* ordering can be defined for the dominant weights. $\lambda > \mu$ if λ, μ are dominant weights and $\mu = \lambda - \alpha$ with α a positive root.

Consider now the Hamiltonian $\mathcal{H}$ written in the variables $z = \mathrm{e}^{\mathrm{i}\theta}$

$$\mathcal{H} = \sum_{i=1}^{N} (z_i \partial_{z_i})^2 + \beta \sum_{i \neq j} (w_{ij} + \bar{w}_{ij}) z_i \partial_{z_i} + \sum_{i=1}^{N} (c_1 w_{i0} + 2c_2 \bar{w}_{ii}) z_i \partial_{z_i} \tag{3.1}$$

with $w_{ij} = (z_i + z_j)/(z_i - z_j)$, $\bar{w}_{ij} = (z_i + z_j^{-1})/(z_i - z_j^{-1})$ and $w_{i0} = (z_i + 1)/(z_i - 1)$. It is possible to construct eigenfunctions of $\mathcal{H}$, polynomial in the variables $z_j^{\pm 1/2} = \mathrm{e}^{\pm \mathrm{i}\theta_j/2}$

and symmetric under the transformations defined by the Weyl group. To achieve that, start from the monomials

$$z_1^{\lambda_1} \dots z_N^{\lambda_N}$$

and define the symmetrized monomials

$$m_\lambda = \sum_{s \in W} z_1^{s(\lambda_1)} \dots z_N^{s(\lambda_N)} \tag{3.2}$$

where the sum is over the elements of the Weyl group W, each distinct monomial occurring only once, and λ denotes a dominant weight of the root system under consideration. The Hamiltonian $\mathcal{H}$ is triangular on the basis of symmetrized monomials m_λ

$$\mathcal{H} m_\lambda = E_\lambda m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu .$$

One can check that there are no poles generated by the w_{ij} factors. The symmetry of m_λ ensures that these poles disappear and lower order monomials are generated. The typical example is

$$\frac{z_1 + z_2}{z_1 - z_2} (z_1^{n_1} z_2^{n_2} - z_1^{n_2} z_2^{n_1}) = z_1^{n_1} z_2^{n_2} + 2 z_1^{n_1 - 1} z_2^{n_2 + 1} + \dots + z_1^{n_2} z_2^{n_1}$$

where $n_1 - n_2$ is a positive integer. The terms containing w_{ij}, $\bar{w}_{ij}$, w_{i0} and $\bar{w}_{ii}$ generate a lower-order symmetric monomial of the type $m_{\lambda - \alpha}$ with α equal to $e_i - e_j$, $e_i + e_j$, e_i and $2e_i$, respectively.

As $\mathcal{H}$ is triangular, the eigenvalues E_λ are easily derived

$$E_\lambda = \sum_{i=1}^{N} \lambda_i (\lambda_i + 2\beta(N - i) + c_1 + 2c_2). \tag{3.3}$$

The restrictions on the values of the momenta λ_i, coming from the fact that λ is a dominant weight, are discussed in the appendices.

4. Jacobi polynomials

The polynomial eigenfunction of $\mathcal{H}$ with BC_N symmetry already considered are also symmetric polynomials in the variables $x_j = \cos\theta_j$. They are multivariate generalizations of the Jacobi polynomials [11].

Let us start with the simplest cases $\beta = 0$ or 1, when the Hamiltonian (2.2) decouples to a sum of one-particle terms H_1. After a gauge transform $\varphi(\theta) H_1 \varphi^{-1}(\theta)$, with $\varphi(\theta) = \sin^{c_1}(\theta/2) \sin^{c_2}\theta$, we obtain the one-particle Hamiltonian

$$\mathcal{H}_1 = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} - \left(c_1 \operatorname{ctg} \frac{\theta}{2} + 2c_2 \operatorname{ctg}\theta \right) \frac{\mathrm{d}}{\mathrm{d}\theta}. \tag{4.1}$$

The eigenfunctions of $\mathcal{H}_1$ satisfy the hypergeometric differential equation in the variable $x = \cos\theta$

$$(1 - x^2) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - [c_1 + (c_1 + 2c_2 + 1)x] \frac{\mathrm{d}y}{\mathrm{d}x} + n(n + c_1 + 2c_2) y = 0. \tag{4.2}$$

The Jacobi polynomials $P_n^{(a,b)}(x)$, with

$$a = c_1 + c_2 - 1/2 \qquad b = c_2 - 1/2 \tag{4.3}$$

and n a non-negative integer, are solutions of this equation. We will use a, b to index the wavefunctions and continue to use c_1, c_2 as coupling constants in the Hamiltonian.

The Jacobi polynomials form a basis for the functions defined on the interval $[-1, 1]$, orthogonal with respect to the scalar product

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} \mathrm{d}x (1-x)^{a-b}(1-x^2)^b f(x)g(x). \tag{4.4}$$

The second (non-polynomial) solution of (4.2) is given by the Jacobi's function of the second kind, $Q_n^{a,b}(x)$. For a detailed description of the Jacobi polynomials and Jacobi functions see [12]. We retain the following expansion property

$$\sum_{n=0}^{\infty}\{h_n^{(a,b)}\}^{-1} P_n^{(a,b)}(x) Q_n^{(a,b)}(y) = \frac{1}{2}\frac{(y-1)^{-a+b}(y^2-1)^{-b}}{y-x} \tag{4.5}$$

where $h_n^{(a,b)}$ is the norm of the Jacobi polynomials with respect to the scalar product (4.4) and $x = \cos\theta$ and $y = \cos\phi$.

The Jacobi polynomials play the same role for the BC_N model as the power functions z^n do for the periodic model. In particular, bosonic (fermionic) wavefunctions can be obtained by symmetrization (antisymmetrization) of products of Jacobi polynomials

$$\mathcal{J}_{\lambda_1,\ldots\lambda_N}^{(a,b)}(\cos\theta_1,\ldots,\cos\theta_N;0) = d_\lambda(0,a,b)\sum_{\sigma\in S_N} P_{\lambda_1}^{(a,b)}(\cos\theta_{\sigma_1})\ldots P_{\lambda_N}^{(a,b)}(\cos\theta_{\sigma_N}) \tag{4.6}$$

and respectively

$$\mathcal{J}_{\lambda_1,\ldots\lambda_N}^{(a,b)}(\cos\theta_1,\ldots,\cos\theta_N;1) = d_\lambda(1,a,b)\frac{\det(P_{\lambda_i+N-i}^{(a,b)}(\cos\theta_j))}{\prod_{i<j}\sin((\theta_i-\theta_j)/2)\sin((\theta_i+\theta_j)/2)} \tag{4.7}$$

where $d_\lambda(\beta, a, b)$ are normalization constants to be fixed later.

The functions defined by the relation (4.7) are analogous to the Schur polynomials. In the particular case $c_1 = 0, c_2 = 1$ (or $a = b = 1/2$) they are, up to a normalization constant, the characters of the symplectic group [13].

For a generic value of β, Lassalle [11] showed that there are eigenfunctions of $\mathcal{H}$, uniques up to a normalization, which have a triangular expansion on the Jack polynomials

$$\mathcal{J}_\lambda^{(a,b)}(x.;\beta) = \sum_{\mu\subseteq\lambda} c_{\lambda\mu} J_\mu(x.;\beta) \tag{4.8}$$

where $\mu \subseteq \lambda$ means $\mu_i \leqslant \lambda_i$ for all i. They were named generalized Jacobi polynomials. We choose their normalization such that $c_{\lambda\lambda} = 1$. Here we have used a result of [14], section 3, to relate the Jack polynomials in the variables $x = \cos\theta$ to the ones in the variables $\sin^2\theta/2 = (1-x)/2$ used in [11].

A method to express these polynomials in terms of Jack polynomials (associated with the A_{N-1} root systems) was also proposed in [15], using a bosonic representation of the Calogero–Sutherland Hamiltonian.

5. Duality

It was proven by Macdonald [2] and by Gaudin [7] that the eigenfunctions of the periodic model for two different coupling constants (β and $1/\beta$) are in correspondence. We show that a similar property holds for the BC_N model. Since the method of [2] is difficult to parallel in the BC_N case, we employ the method proposed by Gaudin.

Consider two sets of independent variables $\theta. = \{\theta_i, i = 1, N\}$ and $\phi. = \{\phi_m, m = 1, M\}$ and the kernel

$$K_{NM}(\theta.;\phi.) = \prod_{m=1}^{M}\prod_{i=1}^{N} \sin\left(\frac{\theta_i - \phi_m}{2}\right)\sin\left(\frac{\theta_i + \phi_m}{2}\right). \tag{5.1}$$

This kernel intertwines between the Hamiltonians $\mathcal{H}_N(\theta.;\beta, c_1, c_2)$ and $\mathcal{H}_M(\phi.; 1/\beta, \tilde{c}_1, \tilde{c}_2)$

$$\begin{aligned} -\beta^{-1/2}[\mathcal{H}_N(\theta.;\beta,c_1,c_2) - \beta MN(2N-1)/2 - MN(c_1+2c_2)]K_{NM}(\theta.;\phi.) \\ = \beta^{1/2}[\mathcal{H}_M(\phi.;1/\beta,\tilde{c}_1,\tilde{c}_2) - \beta^{-1}NM(2M-1)/2 \\ -MN(\tilde{c}_1+2\tilde{c}_2)]K_{NM}(\theta.;\phi.) \end{aligned} \tag{5.2}$$

where the dual values $\tilde{c}_1$ and $\tilde{c}_2$ are defined by

$$\tilde{c}_1 = c_1/\beta \qquad \tilde{c}_2 = (2c_2 - \beta + 1)/2\beta. \tag{5.3}$$

The proof uses repeatedly the identity $\operatorname{ctg} x \operatorname{ctg} y + \operatorname{ctg} y \operatorname{ctg} z + \operatorname{ctg} z \operatorname{ctg} x = 1$ for angles satisfying $x + y + z = 0$.

Let us take first $M = 1$ and evaluate the action of the kinetic operator: $\partial_\theta^2 = \sum_i \partial_i^2$ on $K_N = K_{N1}$

$$\partial_i K_N = \frac{1}{2}\left[\operatorname{ctg}\left(\frac{\theta_i - \phi}{2}\right) + \operatorname{ctg}\left(\frac{\theta_i + \phi}{2}\right)\right] K_N \tag{5.4}$$

$$\partial_\theta^2 K_N = \left[-\frac{N}{2} + \frac{1}{2}\sum_i \operatorname{ctg}\left(\frac{\theta_i - \phi}{2}\right)\operatorname{ctg}\left(\frac{\theta_i + \phi}{2}\right)\right] K_N. \tag{5.5}$$

Using the property $\operatorname{ctg} x \operatorname{ctg} y + \operatorname{ctg} y \operatorname{ctg} z + \operatorname{ctg} z \operatorname{ctg} x = 1$ for the angles $x = (\theta_i - \phi)/2$, $y = -(\theta_i + \phi)/2$ and $z = \phi$, we obtain

$$\operatorname{ctg}\left(\frac{\theta_i - \phi}{2}\right)\operatorname{ctg}\left(\frac{\theta_i + \phi}{2}\right) K_N = (-2\operatorname{ctg}\phi\,\partial_\phi - 1)K_N \tag{5.6}$$

so the kinetic term is

$$\partial_\theta^2 K_N = -(N + \operatorname{ctg}\phi\,\partial_\phi)K_N. \tag{5.7}$$

The one-body part of the potential in the variables θ can be transformed into a derivative acting on the variable ϕ

$$-\sum_i \left(c_1 \operatorname{ctg}\frac{\theta_i}{2} + 2c_2 \operatorname{ctg}\theta_i\right)\partial_i K_N = \left[\left(c_1 \operatorname{ctg}\frac{\phi}{2} + 2c_2\operatorname{ctg}\phi\right)\partial_\phi + c_1 + 2c_2\right]K_N. \tag{5.8}$$

The two-body part of the potential in the variables θ reconstitutes the kinetic part of the Hamiltonian in the variable ϕ

$$\begin{aligned} -\beta\sum_{i<j}\left[\operatorname{ctg}\left(\frac{\theta_i - \theta_j}{2}\right)(\partial_{\theta_i} - \partial_{\theta_j}) + \operatorname{ctg}\left(\frac{\theta_i + \theta_j}{2}\right)(\partial_{\theta_i} + \partial_{\theta_j})\right]K_N \\ = \beta N(N-1)K_N - \frac{\beta}{2}\sum_{i<j}\left(C_{i+}C_{j+} + C_{i-}C_{j-} - C_{i-}C_{j+} - C_{i+}C_{j-}\right)K_N \\ = \beta N(N-1)K_N + \frac{\beta}{4}\left[\left(\sum_i (C_{i-} - C_{i+})\right)^2 - \sum_i (C_{i-} - C_{i+})^2\right]K_N \\ = \beta\left[N\left(N - \frac{1}{2}\right) + \partial_\phi^2 + \tfrac{1}{2}\operatorname{ctg}\left(\frac{\theta_i - \phi}{2}\right)\operatorname{ctg}\left(\frac{\theta_i + \phi}{2}\right)\right]K_N \end{aligned} \tag{5.9}$$

where $C_{i\pm} = \text{ctg}((\theta_i \pm \phi)/2)$. Using (5.7), (5.8) and (5.9) we obtain

$$\mathcal{H}(\theta; \beta, c_1, c_2)K_N = \beta\left[\partial_\phi^2 + \left(\tilde{c}_1 \,\text{ctg}\,\frac{\phi}{2} + 2\tilde{c}_2 \,\text{ctg}\,\phi\right)\partial_\phi\right]K_N + (\beta N^2 + (c_1 + 2c_2 - \beta + 1)N)K_N. \tag{5.10}$$

The right-hand side of this equation is, up to a constant, the Hamiltonian for one particle of coordinate ϕ, with the new coupling constants $\tilde{c}_1 = c_1/\beta$ and $\tilde{c}_2 = (2c_2 - \beta + 1)/2\beta$.

Let us now take M variables ϕ, $M > 1$ and $K_{NM}(\theta_1, \ldots, \theta_N; \phi_1, \ldots, \phi_M) = \prod_{i=1}^M K_N(\theta_1, \ldots, \theta_N; \phi_i)$. The potential part of the Hamiltonian $\mathcal{H}(\theta; \beta, c_1, c_2)$ is a first-order derivative, so its action on the kernel $K_{NM}(\phi_1, \ldots, \phi_M)$ is additive. The second-order derivatives in the kinetic energy operator generate crossed terms which correspond to the two-body terms of the Hamiltonian in the variables $\phi_1, \ldots, \phi_M$:

$$\begin{aligned}\partial_\theta^2 K_{NM} &= \sum_{i=1}^N\left[-\frac{M}{2} + \frac{1}{4}\left(\sum_{m=1}^M (C_{im-} + C_{im+})\right)^2 - \frac{1}{4}\sum_{m=1}^M (C_{im-}^2 + C_{im+}^2)\right]K_{NM} \\ &= \left[-\frac{NM}{2} + \frac{1}{2}\sum_{i,m}\text{ctg}\,\frac{\theta_i - \phi_m}{2}\,\text{ctg}\,\frac{\theta_i + \phi_m}{2}\right. \\ &\quad\left. + \sum_{m\neq n}\left(\text{ctg}\,\frac{\phi_m - \phi_n}{2} + \text{ctg}\,\frac{\phi_m + \phi_n}{2}\right)\partial_{\phi_m}\right]K_{NM}.\end{aligned} \tag{5.11}$$

Here we have used a calculation of the same type as in (5.9), but involving the index m of $C_{im\pm} = \text{ctg}((\theta_i \pm \phi_m)/2)$ instead of the index i. The last term in (5.11) is proportional to the two-body interaction in variables ϕ.

The full result is obtained by collecting the partial results in (5.9), (5.6) (summed over the M variables ϕ_m) and (5.11):

$$\mathcal{H}_N(\theta.; \beta, c_1, c_2)K_{NM}(\theta.; \phi.) = [-\beta\mathcal{H}_M(\phi.; 1/\beta, \tilde{c}_1, \tilde{c}_2) + \beta MN(N-1) + NM^2 + MN(c_1 + 2c_2)]K_{NM}(\theta.; \phi.). \tag{5.12}$$

This is equivalent to the result announced in equation (5.2). In the next section, this property will be used in order to obtain the expansion of $K(\theta.; \phi.)$ in terms of the eigenfunctions of $\mathcal{H}_N(\theta.; \beta, c_1, c_2)$ and $\mathcal{H}_M(\phi.; 1/\beta, \tilde{c}_1, \tilde{c}_2)$.

Using a similar method, we can derive another result whose periodic analogue is well known [2, 3]. It concerns the expansion of $K^{-\beta}(\theta.; \phi.)$ on the eigenfunctions of $\mathcal{H}(\theta; \beta, c_1, c_2)$

$$\begin{aligned}&[\mathcal{H}_N(\theta.; \beta, c_1, c_2) - \mathcal{H}_M(\phi.; \beta, -c_1, -c_2 + \beta)]K_{NM}^{-\beta}(\theta.; \phi.) \\ &\quad = -[\beta^2 MN(M - N + 1) - \beta MN(c_1 + 2c_2)]K_{NM}^{-\beta}(\theta.; \phi.).\end{aligned} \tag{5.13}$$

We can further transform this expression by noting the following property

$$\psi^{-1}(\phi.)\mathcal{H}_M(\phi.; \beta, c_1, c_2)\psi(\phi.) = \mathcal{H}_M(\phi.; \beta, -c_1, -c_2 + 1) - C_1 \tag{5.14}$$

where $\psi(\phi.) = \prod_{i=1}^M(\sin^{-2c_1}(\phi_i/2)\sin^{-2c_2+1}\phi_i)$ and $C_1 = M(c_1 + 2c_2 - 1)(\beta(M-1) + 1)$. This allows us to rewrite (5.13) as

$$[\mathcal{H}_N(\theta.; \beta, c_1, c_2) - \mathcal{H}_M(\phi.; \beta, c_1, c_2 - \beta + 1) + C_2]\psi(\phi.)K_{NM}^{-\beta}(\theta.; \phi.) = 0 \tag{5.15}$$

where the constant

$$C_2 = \beta^2 MN(M - N + 1) - (c_1 + 2c_2)\beta MN + M(c_1 + 2c_2 - 2\beta + 1)(\beta(M-1) + 1).$$

6. Expansion formula for $K_{NM}(\theta.; \phi.)$

The kernel (5.1) is a polynomial in both sets of variables $y_i = \cos\theta_i$ and $w_m = \cos\phi_m$

$$K_{NM}(\theta.; \phi.) = \prod_{m=1}^{M}\prod_{i=1}^{N} \sin\left(\frac{\theta_i - \phi_m}{2}\right)\sin\left(\frac{\theta_i + \phi_m}{2}\right) = 2^{-NM}\prod_{m=1}^{M}\prod_{i=1}^{N}(y_i - w_m). \tag{6.1}$$

It plays the role of a generating function for the generalized Jacobi polynomials. Using equation (5.2) we prove the following property, similar to the dual expansion of the Jack polynomials [2, 3]

$$K_{NM}(\theta.; \phi.) = 2^{-NM}\sum_{\lambda}(-1)^{|\tilde{\lambda}|}\mathcal{J}_{\lambda}^{(a,b)}(y.; \beta)\mathcal{J}_{\tilde{\lambda}}^{(\tilde{a},\tilde{b})}(w.; 1/\beta) \tag{6.2}$$

where

$$a = c_1 + c_2 - 1/2 \qquad b = c_2 - 1/2 \qquad \tilde{a} = (a - \beta + 1)/\beta \qquad \tilde{b} = (b - \beta + 1)/\beta$$

and the symbol $\tilde{\lambda}$ denotes the partition with parts $\tilde{\lambda}_k = N - \lambda'_{M-k+1}$, where λ' denotes the partition conjugate to λ.

Let us prove this relation. Call $A_\lambda(w.; \beta, a, b)$ the coefficients of the expansion of $K_{NM}(\theta.; \phi.)$ on eigenfunctions of $\mathcal{H}_N(\theta.; \beta, c_1, c_2)$,

$$K_{NM}(\theta.; \phi.) = 2^{-NM}\sum_{\lambda} A_\lambda(w.; \beta, a, b)\mathcal{J}_{\lambda}^{(a,b)}(y.; \beta). \tag{6.3}$$

It follows from equation (5.2) that they are eigenfunctions of $\mathcal{H}_M(\phi.; 1/\beta, \tilde{c}_1, \tilde{c}_2)$, corresponding to the same eigenvalue as $\mathcal{J}_{\tilde{\lambda}}^{(\tilde{a},\tilde{b})}(w.; 1/\beta)$. As the energy levels can be degenerate, we still have to prove that these $A_\lambda(w.; \beta, a, b)$ are proportional to $\mathcal{J}_{\tilde{\lambda}}^{(\tilde{a},\tilde{b})}(w.; 1/\beta)$ and to determine the proportionality constant. To prove this one can use the duality property of the Jack polynomial and exploit their relation to the generalized Jacobi polynomials (4.8).

Expression (6.1) can be expanded on the Jack polynomials [3]

$$K_{NM}(\theta.; \phi.) = 2^{-NM}\sum_{\lambda}(-1)^{|\tilde{\lambda}|}J_\lambda(y.; \beta)J_{\tilde{\lambda}}(w.; 1/\beta) \tag{6.4}$$

where we have used the relation between the Jack polynomials with arguments w and with arguments w^{-1}

$$J_{\tilde{\lambda}}(w.; 1/\beta) = \prod_{m=1}^{M} w_m^N J_{\lambda'}(w^{-1}.; 1/\beta).$$

This property can easily be verified using the triangularity of $J_\lambda(y; \beta)$ in the basis of symmetric monomials m_λ and the fact that boths sides are eigenfunctions of the periodic Calogero–Sutherland corresponding to the same eigenvalue.

In equation (6.3) we can expand the generalized Jacobi polynomials on the Jack polynomials, to obtain

$$K_{NM}(\theta.; \phi.) = 2^{-NM}\sum_{\lambda} A_\lambda(w.; \beta, a, b)\sum_{\mu\subseteq\lambda} c_{\lambda,\mu}J_\mu(y.; \beta) \tag{6.5}$$

where $\mu \subseteq \lambda$ means $\mu_i \leqslant \lambda_i$ for all i. From the relation (6.4) and the orthogonality of Jack polynomials, we have

$$J_{\tilde{\mu}}(w.; 1/\beta) = (-1)^{|\mu|}\sum_{\lambda\subseteq\mu} c_{\lambda\mu}A_\lambda(w.; \beta, a, b). \tag{6.6}$$

Inverting this expansion we obtain

$$A_\lambda(w.;\beta,a,b)=\sum_{\tilde\mu\subseteq\tilde\lambda}(-1)^{|\mu|}c'_{\lambda\mu}J_{\tilde\mu}(w.;1/\beta) \tag{6.7}$$

with $c'_{\lambda\lambda}=1$. As the generalized Jacobi polynomials $\mathcal{J}_{\tilde\lambda}^{(\tilde a,\tilde b)}(w.;1/\beta)$ are uniquely defined as being eigenfunctions of the Hamiltonian $\mathcal{H}_M(\phi.;1/\beta,\tilde c_1,\tilde c_2)$ which is triangular in the basis of Jack polynomials and with the coefficient $c_{\lambda\lambda}=1$, we conclude that

$$A_\lambda(w.;\beta,a,b)=(-1)^{|\tilde\lambda|}\mathcal{J}_{\tilde\lambda}^{(\tilde a,\tilde b)}(w.;1/\beta)$$

which proves (6.2). Note that the expansion in (6.2) contains just a finite number of terms, corresponding to partitions λ included in the partition M^N.

The equations (5.13) and (5.15) can also be related to expansion relations for $K_{NM}^{-\beta}(\theta.;\phi.)$ and $\psi(\phi.)K_{NM}^{-\beta}(\theta.;\phi.)$, probably involving generalized hypergeometric functions which are not polynomials. The simplest case of (5.15) is $\beta=N=M=1$, when the associate expansion relation is the expansion property of the Jacobi functions (4.5).

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Appendix A. Root systems

The main characteristics of the root systems D_N, B_N, C_N and BC_N are the following.

(i) *The D_N root system.* The positive roots are

$$e_i-e_j,\, e_i+e_j \qquad 1\leqslant i<j\leqslant N.$$

The fundamental weights are

$$\bar\omega_i=e_1+\cdots+e_i \qquad 1\leqslant i\leqslant N-2$$
$$\bar\omega_{N-1}=\tfrac{1}{2}(e_1+\cdots+e_{N-2}+e_{N-2}-e_N)$$
$$\bar\omega_N=\tfrac{1}{2}(e_1+\cdots+e_{N-2}+e_{N-2}+e_N).$$

The dominant weights of D_N are indexed by $\lambda_1\geqslant\cdots\geqslant|\lambda_N|\geqslant 0$ all integers or all half-integers. λ_N can be positive or negative. The action of the Weyl group on λ_i is generated by the permutations $s_{ij}\lambda_i=\lambda_j$ and by $\bar s_{ij}\lambda_i=-\lambda_j$.

(ii) *The C_N root system.* The positive roots are

$$e_i-e_j,\ e_i+e_j \qquad 1\leqslant i<j\leqslant N \qquad 2e_i \qquad (1\leqslant i\leqslant N).$$

The fundamental weights are

$$\bar\omega_i=e_1+\cdots+e_i \qquad 1\leqslant i\leqslant N$$

and the dominant weights are characterized by $\lambda_1\geqslant\cdots\geqslant\lambda_N\geqslant 0$ a set of positive (or zero) integers. The Weyl group contains, beside the permutations s_{ij} and $\bar s_{ij}$, the reflections $s_i(\lambda_i)=-\lambda_i$.

(iii) *The B_N root system.* The positive roots are

$$e_i-e_j,\ e_i+e_j \qquad 1\leqslant i<j\leqslant N \qquad e_i \qquad 1\leqslant i\leqslant N.$$

The fundamental weights are

$$\bar{\omega}_i = e_1 + \cdots + e_i \qquad 1 \leqslant i \leqslant N-1$$
$$\bar{\omega}_N = \tfrac{1}{2}(e_1 + \cdots + e_{N-1} + e_N)$$

and the dominant weights are characterized by $\lambda_1 \geqslant \cdots \geqslant \lambda_N \geqslant 0$ all integers or all half-integers. The Weyl group is that of C_N.

(iv) *The BC_N root system.* Its positive roots are

$$e_i - e_j,\ e_i + e_j \qquad 1 \leqslant i < j \leqslant N \qquad e_i,\ 2e_i \qquad 1 \leqslant i \leqslant N.$$

The Weyl group and the dominant weights are those of C_N.

Appendix B. Some examples of eigenfunctions

In this appendix we give as an example some eigenvectors of $\mathcal{H}$ at $N = 2$. These examples illustrate the 'selection rules' of the previous appendix, imposed by the different symmetries on the momenta λ_i. The following remarks are valid for any N:
—polynomials labelled by half-integer weights (all $\lambda_i \in \mathbb{Z} + 1/2$) are allowed only for the B_N and D_N cases
—for D_N, $\lambda_N \neq 0$, the levels are doubly degenerate, $E_{\lambda_1,\ldots,\lambda_{N-1},\lambda_N} = E_{\lambda_1,\ldots,\lambda_{N-1},-\lambda_N}$.

BC_2

$$\mathcal{J}_{1,0} = m_{1,0} + \frac{4c_1}{1+2\beta+c_1+2c_2} m_{0,0}$$
$$\mathcal{J}_{1,1} = m_{1,1} + \frac{2c_1}{1+c_1+2c_2} m_{1,0} + \frac{4c_1^2 + 4\beta(1+c_1+2c_2)}{(1+\beta+c_1+2c_2)(1+c_1+2c_2)} m_{0,0} \qquad \text{(B1)}$$

$C_2(c_1 = 0)$

$$P_{1,0} = m_{1,0}$$
$$P_{1,1} = m_{1,1} + \frac{4c_1^2 + 4\beta(1+2c_2)}{(1+\beta+2c_2)(1+2c_2)} m_{0,0} \qquad \text{(B2)}$$

$B_2(c_2 = 0)$

$$P_{1/2,1/2} = m_{1/2,1/2}$$
$$P_{1,0} = m_{1,0} + \frac{4c_1}{1+2\beta+c_1} m_{0,0}$$
$$P_{3/2,1/2} = m_{3/2,1/2} + \frac{4\beta+6c_1}{2+c_1+2\beta} m_{1/2,1/2} \qquad \text{(B3)}$$

$D_2(c_1 = c_2 = 0)$

$$P_{1/2,\pm 1/2} = m_{1/2,\pm 1/2}$$
$$P_{1,0} = m_{1,0}$$
$$P_{1,\pm 1} = m_{1,\pm 1} + \frac{2\beta}{\beta+1} m_{0,0}$$
$$P_{3/2,\pm 1/2} = m_{3/2,\pm 1/2} + \frac{2\beta}{\beta+1} m_{1/2,\mp 1/2}. \qquad \text{(B4)}$$

The symmetric monomials m_{λ_1,λ_2} associated with each type of symmetry were defined in (3.2).

Unlike in the case of Jack polynomials, there is an N-dependence of the coefficients of m_λ.

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